TRINITY COLLEGE

IN NOMINE DOMINI

Semester Two Examination, 2022

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1&2

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words ______________________________

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Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	49	35
Section Two: Calculator-assumed	12	12	100	94	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (49 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

(a) Determine a unit vector in the same direction as $\mathbf{a} - 3\mathbf{b}$ when $\mathbf{a} = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$. (3 marks)

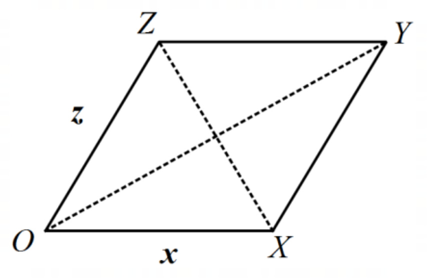
Solution
$\mathbf{a} - 3\mathbf{b} = \begin{pmatrix} -7 \\ 5 \end{pmatrix} - 3\begin{pmatrix} -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \end{pmatrix}$ $\left\lvert\begin{pmatrix} 5 \\ -10 \end{pmatrix}\right\rvert = 5\sqrt{1^2 + 2^2} = 5\sqrt{5}$ $u = \dfrac{1}{5\sqrt{5}}\begin{pmatrix} 5 \\ -10 \end{pmatrix} = \dfrac{\sqrt{5}}{5}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$
Specific behaviours
✓ calculates $3\mathbf{a} - \mathbf{b}$ ✓ calculates magnitude   ✓ correct unit vector

(b) Determine the value(s) of the constant μ so that the vectors $\begin{pmatrix} \mu + 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} \mu \\ \mu + 3 \end{pmatrix}$ are perpendicular. (3 marks)

Solution
$\begin{pmatrix} \mu + 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} \mu \\ \mu + 3 \end{pmatrix} = 0$ $\mu(\mu + 1) - 2(\mu + 3) = 0$ $\mu^2 - \mu - 6 = 0$ $(\mu + 2)(\mu - 3) = 0$ $\mu = -2, \qquad \mu = 3$
Specific behaviours
✓ equates scalar product to 0   ✓ simplifies scalar product   ✓ states both values

Question 2 **(5 marks)**

Use a vector method to prove that the diagonals OY and XZ of parallelogram $OXYZ$ intersect at right angles if and only if the parallelogram is a rhombus. Let $\overrightarrow{OX} = \mathbf{x}$ and $\overrightarrow{OZ} = \mathbf{z}$.



Solution

For diagonals OY and XZ to intersect at right angles then $\overrightarrow{OY} \cdot \overrightarrow{XZ} = 0$.

Vectors for diagonals are $\overrightarrow{OY} = \mathbf{z} + \mathbf{x}$ and $\overrightarrow{XZ} = \mathbf{z} - \mathbf{x}$.

Scalar product of diagonals:

$$\begin{aligned}\overrightarrow{OY} \cdot \overrightarrow{XZ} &= (\mathbf{z} + \mathbf{x}) \cdot (\mathbf{z} - \mathbf{x}) \\ &= \mathbf{z} \cdot \mathbf{z} - \mathbf{z} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{z} - \mathbf{x} \cdot \mathbf{x} \\ &= \mathbf{z} \cdot \mathbf{z} - \mathbf{x} \cdot \mathbf{x} \\ &= |\mathbf{z}|^2 - |\mathbf{x}|^2\end{aligned}$$

Hence for $|\mathbf{z}|^2 - |\mathbf{x}|^2 = 0$ we require

$$|\mathbf{z}|^2 = |\mathbf{x}|^2$$
$$|\mathbf{z}| = |\mathbf{x}|$$

Hence the diagonals of a parallelogram will only intersect at right angles if all side lengths are equal - the parallelogram is a rhombus.

Specific behaviours

✓ states scalar product of $\overrightarrow{OY}$ and $\overrightarrow{XZ}$ must be zero to intersect at right angles
✓ vectors for diagonals $\overrightarrow{OY}$ and $\overrightarrow{XZ}$ in terms of $\mathbf{x}$ and $\mathbf{z}$
✓ forms and simplifies scalar product of diagonals
✓ shows that side lengths must be equal
✓ explains result

Question 3 **(8 marks)**

(a) Express $\sqrt{3}\cos\theta + \sin\theta$ in the form $R\cos(\theta - \beta)$, where $R > 0$ and $0 < \beta < \frac{\pi}{2}$. (3 marks)

Solution
$R\cos(\theta-\beta) = R\cos\theta\cos\beta + R\sin\theta\sin\beta = \sqrt{3}\cos\theta + \sin\theta$ Comparing coefficients, $R\cos\beta = \sqrt{3}$ and $R\sin\beta = 1$.   Hence $R^2 = 1 + 3 = 4 \to R = 2$ and $\tan\beta = {}^{1}/_{\sqrt{3}} \to \beta = \frac{\pi}{6}$. $\sqrt{3}\cos\theta + \sin\theta = 2\cos\left(\theta - \frac{\pi}{6}\right)$
Specific behaviours
✓ uses sum identity to compare coefficients   ✓ correct value of β ✓ correctly expresses in required form

(b) Determine the value of $\tan\alpha$, where $90° < \alpha < 180°$, given that α satisfies the equation $5\cot^2\alpha - \mathrm{cosec}^2\alpha = 4\,\mathrm{cosec}\,\alpha - 2$. (5 marks)

Solution
$5\cot^2\alpha - \mathrm{cosec}^2\alpha = 4\,\mathrm{cosec}\,\alpha - 2$ $5(\mathrm{cosec}^2\alpha - 1) - \mathrm{cosec}^2\alpha = 4\,\mathrm{cosec}\,\alpha - 2$ $4\,\mathrm{cosec}^2\alpha - 4\,\mathrm{cosec}\,\alpha - 3 = 0$ $(2\,\mathrm{cosec}\,\alpha + 1)(2\,\mathrm{cosec}\,\alpha - 3) = 0$ $2\,\mathrm{cosec}\,\alpha = -1 \to \text{No solution}, \quad 2\,\mathrm{cosec}\,\alpha = 3 \to \mathrm{cosec}\,\alpha = \frac{3}{2}$ In second quadrant $\tan\alpha < 0$: $\cot\alpha = -\sqrt{\left(\frac{3}{2}\right)^2 - 1} = -\frac{\sqrt{5}}{2}$ Hence $\tan\alpha = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$
Specific behaviours
✓ uses identity to eliminate $\cot$ term   ✓ simplifies into quadratic equation   ✓ factors quadratic equation and obtains value for $\mathrm{cosec}\,\alpha$ ✓ uses identity to obtain value for $\cot\alpha$ ✓ correct value for $\tan\alpha$

Question 4 **(8 marks)**

(a) Express $z^2 - 4z + 15$ as the product of its linear factors. (3 marks)

Solution
$(z-2)^2 - 2^2 + 15 = 0$ $(z-2)^2 = -11$ $(z-2)^2 = 11i^2$ $z - 2 = \pm\sqrt{11}i$ $z = 2 \pm \sqrt{11}i$ Hence $z^2 - 4z + 15 = \left(z - 2 + \sqrt{11}i\right)\left(z - 2 - \sqrt{11}i\right)$.
Specific behaviours
✓ completes square ✓ correct roots ✓ writes expression in factored form

(b) Determine the complex numbers u and v given that $u + 2v = 6$ and $u + iv = -1 + 6i$. (5 marks)

Solution
$u + 2v = 6$ $u + iv = -1 + 6i$ $2v - iv = 7 - 6i$ $v(2 - i) = 7 - 6i$ $v = \dfrac{7-6i}{2-i}$ $v = \dfrac{7-6i}{2-i} \cdot \dfrac{2+i}{2+i}$ $= \dfrac{14 + 7i - 12i + 6}{5}$ $= 4 - i$ $u = 6 - 2v$ $= 6 - 2(4 - i)$ $= -2 + 2i$
Specific behaviours
✓ subtracts equations to eliminate u ✓ expresses v as quotient ✓ correct complex number v ✓ substitutes for u ✓ correct complex number u

Question 5 **(8 marks)**

(a) Let $\mathbf{P} = \begin{bmatrix} -2 & 4 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix}$. Determine $3\mathbf{P} - 4\mathbf{I} + \mathbf{PQ}$. (3 marks)

Solution
$$\mathbf{PQ} = \begin{bmatrix} -2 & 4 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -22 & 6 \\ 3 & -2 \end{bmatrix}$$ $$3\mathbf{P} - 4\mathbf{I} + \mathbf{PQ} = \begin{bmatrix} -6 & 12 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} -22 & 6 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -32 & 18 \\ 3 & -9 \end{bmatrix}$$
Specific behaviours
✓ correct scalar multiple of $\mathbf{P}$ ✓ correct product $\mathbf{PQ}$ ✓ correct result

(b) Determine the value(s) of the constant t for which the matrix $\begin{bmatrix} t+1 & 2 \\ 5 & t-2 \end{bmatrix}$ is not singular. (2 marks)

Solution
Require $(t+1)(t-2) - 10 \neq 0$. $t^2 - t - 12 \neq 0$ $(t+3)(t-4) \neq 0$ Hence matrix invertible for $t \in \mathbb{R},\ t \neq -3,\ t \neq 4$.
Specific behaviours
✓ calculates determinant ✓ states correct restrictions on t

(c) Show use of a matrix method to solve the system of equations $10x + 6y + 3 = 0$ and $3x + 2y - 1 = 0$. (3 marks)

Solution
$$\begin{bmatrix} 10 & 6 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$ $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 3 & 2 \end{bmatrix}^{-1}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -6 \\ -3 & 10 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 9.5 \end{bmatrix}$$
Specific behaviours
✓ writes equation in matrix form ✓ obtains inverse matrix ✓ correct solution

Question 6 **(6 marks)**

Recall the property $m = \tan\theta$ for the gradient of a straight line making an angle of θ with the positive x-axis, $0 \le \theta < 90°$ or $90° < \theta \le 180°$.

(a) Show that $\dfrac{2m}{1+m^2} = \sin 2\theta$ and that $\dfrac{1-m^2}{1+m^2} = \cos 2\theta$. (4 marks)

Solution

$$\begin{aligned}\frac{2m}{1+m^2} &= \frac{2\tan\theta}{1+\tan^2\theta}\\ &= \frac{2\tan\theta}{\sec^2\theta}\\ &= \frac{2\sin\theta}{\cos\theta}\times\cos^2\theta\\ &= 2\sin\theta\cos\theta\\ &= \sin 2\theta\end{aligned}$$

$$\begin{aligned}\frac{1-m^2}{1+m^2} &= \frac{1-\tan^2\theta}{1+\tan^2\theta}\\ &= \frac{1-\tan^2\theta}{\sec^2\theta}\\ &= \left(1-\frac{\sin^2\theta}{\cos^2\theta}\right)\times\cos^2\theta\\ &= \cos^2\theta-\sin^2\theta\\ &= \cos 2\theta\end{aligned}$$

Specific behaviours

✓ replaces m with $\tan\theta$ and eliminates $\tan\theta$ for both
✓ uses $1+\tan^2\theta = \sec^2\theta$ identity for both
✓ simplifies and uses double angle identity to obtain $\sin 2\theta$
✓ simplifies and uses double angle identity to obtain $\cos 2\theta$

(b) Hence, or otherwise, determine the image of the point with coordinates $(1, -2)$ when it is reflected in the line $y = 2x$. (2 marks)

Solution

Matrix for reflection in line $y = mx = x\tan\theta$ is

$$\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix}1-m^2 & 2m\\ 2m & m^2-1\end{bmatrix}_{m=2}$$
$$= \frac{1}{5}\begin{bmatrix}-3 & 4\\ 4 & 3\end{bmatrix}$$

$$\frac{1}{5}\begin{bmatrix}-3 & 4\\ 4 & 3\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix} = \begin{bmatrix}-11/5\\ -2/5\end{bmatrix}$$

Coordinates of image: $\left(-11/5, -2/5\right) = (-2.2, -0.4)$.

Specific behaviours

✓ obtains reflection matrix
✓ correct image position vector or coordinates

Question 7 **(8 marks)**

(a) Express the recurring decimal $0.3\overline{45}$ as a simplified rational number. (2 marks)

Solution
$100x = 34.54545\ldots$ $x = 0.34545\ldots$ $99x = 34.2$ $x = \dfrac{342}{990} = \dfrac{19}{55}$
Specific behaviours
✓ indicates appropriate method ✓ correct rational number

(b) Let $f(n) = 3^n + 4^n + 5^n$. Prove by induction that $f(n)$ is divisible by 12 for all positive odd integers n. (6 marks)

Solution

When $n = 1, f(1) = 3 + 4 + 5 = 12$, and so is divisible by 12.

When $n = k$, assume that $f(k)$ is divisible by 12 so that $f(k) = 3^k + 4^k + 5^k = 12I$, where $I \in \mathbb{Z}^+$.

When $n = k + 2$ then

$$\begin{aligned} f(k+2) &= 3^{k+2} + 4^{k+2} + 5^{k+2} \\ &= 9 \cdot 3^k + 16 \cdot 4^k + 25 \cdot 5^k \\ &= 3^k + 4^k + 5^k + 8 \cdot 3^k + 15 \cdot 4^k + 24 \cdot 5^k \\ &= 12I + 8 \cdot 3 \cdot 3^{k-1} + 3 \cdot 5 \cdot 4 \cdot 4^{k-1} + 12 \cdot 2 \cdot 5^k \\ &= 12I + 12 \cdot 2 \cdot 3^{k-1} + 12 \cdot 5 \cdot 4^{k-1} + 12 \cdot 2 \cdot 5^k \\ &= 12(I + 2 \cdot 3^{k-1} + 5 \cdot 4^{k-1} + 2 \cdot 5^{k-1}) \end{aligned}$$

Since $f(n)$ is divisible by 12 when $n = k + 2$, and $f(1)$ is divisible by 12, then $f(n)$ must be divisible by 12 for all positive odd integers n.

Specific behaviours

✓ demonstrates true for $n = 1$
✓ makes assumption for $n = k$
✓ expression for $f(k + 2)$
✓ uses assumption to replace $3^k + 4^k + 5^k$ with $12I$
✓ factors out 12 from $f(k + 2)$
✓ concluding statement

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________